Types of Generalization Made by Pupils Aged 12–13 and by Their Future Mathematics Teachers

Spôsoby zovšeobecňovania u žiakov vo veku 12–13 rokov a ich budúcich učiteľov matematiky

Mária Slavíčková[1]

[1] Katedra didaktiky matematiky, fyziky a informatiky, Fakulta matematiky, fyzika informatiky, Univerzita Komenského v Bratislave, Mlynská dolina, 842 48 Bratislave, Slovak Republic; slavickova@fmph.uniba.sk

This paper seeks to establish what kind of arguments pupils (aged 12–13) use and how they make their assumptions and generalizations. Our research also explored the same phenomenon in the case of graduate mathematics teachers studying for their masters' degrees in our faculty at that time. The main focus was on algebraic reasoning, in particular pattern exploring and expressing regularities in numbers. In this paper, we introduce the necessary concepts and notations used in the study, briefly characterize the theoretical levels of cognitive development and terms from the Theory of Didactical Situations. We set out to answer three research questions. To collect the research data, we worked with a group of 32 pupils aged 12–13 and 19 university students (all prospective mathematics teachers in the first year of their master's). We assigned them two flexible tasks to and asked them to explain their findings/formulas. Besides that, we collected additional (supportive) data using a short questionnaire. The supporting data concerned their opinions on the tasks and the explanations. The results and limited scope of the research indicated what should be changed in preparing future mathematics teachers. These changes could positively influence the pupils' strategies of solving not only flexible tasks but also their ability to generalize.

Key words: generalization, algebraic thinking, functional thinking, looking for patterns.

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Cieľom článku je zistiť, akým spôsobom zovšeobecňujú a akými argumentami svoje zistenia podkladajú žiaci vo veku 12–13 rokov. To isté sme zisťovali aj u budúcich učiteľov matematiky, ktorý v tom čase študovali na našej fakulte. Hlavné zameranie článku je na algebrické uvažovanie, špeciálne na skúmanie vzorov a vyjadrenie zákonitosti pomocou číselných hodnôt. V tomto článku vysvetľujeme dôležité koncepty a značenia použité vo výskume, stručne charakterizujeme úrovne kognitívneho rozvoja a využité pojmy z Teórie didaktických situácii. Stanovili sme si tri výskumné otázky. Na získanie výskumných dát sme pracovali so skupinou 32 žiakov vo veku 12–13 rokov a s 19 študentmi učiteľstva matematiky. Obom skupinám sme zadali rovnaké úlohy na riešenie a požiadali sme ich o vysvetlenie riešenia, resp. vzťahu, ku ktorému sa dopracovali. Okrem toho sme krátkym dotazníkom zozbierali podporné dáta ohľadom skúmaných skupín. Výsledky sú diskutované v kontexte podobných výskumov, sú identifikované limitácie opísaného výskumu a vyvodené závery. Na základe diskusie a limitov sme sformulovali odporučenia pre zmenu v príprave budúcich učiteľov matematiky.

Klíčová slova: proces zovšeobecňovania, algebraické myslenie, funkčné myslenie tvorba vzorcov.

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1 Introduction

We often get used to generalizing in most situations in life. Not all our generalizations are correct, and some are based on one or two preconceptions (like "all men are…"). On the other hand, generalization is also a part of mathematics education and mathematics literacy.

> Generalization is a heartbeat of mathematics and appears in many forms. If teachers are unaware of its presence and are not in the habit of getting students to work at expressing their own generalizations, then mathematical thinking is not taking place. (Mason, 1996, p. 65)

Without generalization and argumentation, mathematics is just a set of rules to memorize. Nevertheless, in talking to the masters students we found out that presenting the rule and then solving "problems" with its application is one of the favorite ways of teaching mathematics in the classroom.

The importance of generalization and argumentation is essential in mathematics lessons, because

> formalizing one's reasoning could lead to practical benefits such as clarity of thoughts and expressions, improved objectivity, and greater confidence. The ability to analyze others' arguments can also serve as a yardstick for when to withdraw from discussions that will most likely be futile. (Almossawi, 2016, p. 4)

If we can argue the correctness of our solution (or defend the solution of our classmate), we take responsibility for our decisions, we do not need an authority to tell us whether we are right or not, and this is an essential ability due to the fact that "in the real world of problem-solving and doing mathematics there are no answer books" (Van de Walle, Karp, & Bay-Williams, 2012, p. 37).

We have focused on algebra, particularly on functional thinking, in our paper. As Lee and Wheeler (1987, p. 44) pointed out, the abstractness of algebra is one reason for students' problems. Algebra, as Kaput (1999, p. 134) stated, has been a route to higher mathematics but, at the same time, a barrier for many students, forcing them to take another educational direction. One of the reasons could be a big step between arithmetic to algebra. In our research, we focus on the pupil's description of calculations and strategies for solving given tasks. Several competences are needed when students are moving from arithmetic, number handling, and computation to algebra. Kaput (1995) identified three such competencies. In this paper, we will only focus on the competence concerning the ability to describe the calculations one wants to do. It can be said in words, by abbreviations, or by a formula.

While studying the research done on generalization (like Russel et al., 2011; Van de Walle et al., 2012; Rivera & Becker, 2005, 2007; Blanton, 2008, etc) we found out that tasks requiring generalization are included in the curriculum in several countries. This is not the case with the Slovak curriculum. Therefore, we were curious whether pupils could do generalization, explanation, and argumentation without being presented with generalizing tasks in their mathematics lessons. What kind of arguments would they use (if any), how would they make their assumptions and generalizations, etc.

2 Theoretical background

Firstly, we introduce the necessary concepts and notations used in our study concerning functional thinking and ways of generalization. Secondly, the theoretical levels of cognitive development and the terms from the Theory of Didactical Situation used in our study are briefly characterized.

2.1 Generalization and reasoning

We will focus on algebraic reasoning which, according to Blanton and Kaput (2005, p. 413), can take various forms, including reasoning about operations and properties associated with numbers (like commutative property), pattern exploring and expressing regularities in numbers, generalizing regularities and operations on classes of objects (usually described as "abstract algebra"). The focus will be on exploring and expressing regularities in growing patterns.

Our study is concerned with a figure series that changes (mostly grows) according to a predictable pattern. The goal is to find relating numbers to growing pattern tasks that are visual (e.g., made with geometric figures) by asking pupils and masters students to discover the regularities involved and develop generalizations for function rules (Friel & Markworth, 2009, p. 27). According to Friel and Markworth (2009, p. 30), analyzing growing patterns should include the developmental progression of reasoning by looking at the visuals, reasoning about the numerical relationships, and then extending to the larger (or n-th) case.

Pupils can express the result in a recursive or explicit formula. By a recursive formula, we mean a form in which the description of the phenomenon is based on the previous figure in a row (for example, $a_n = a_{n-1} + 4$). By an explicit formula, we mean a formula which gives us the result directly (like $a_n = 4n + 2$). These two approaches partly define two groups of generalizators, as Becker and Rivera (2006, p. 446) describe. The first group, predominantly numerical generalizators, set up tables of values when solving growing patterns tasks. As regards solving, they are looking at numbers and trying to figure out the solution. In many cases, variables are only placeholders and by working with numerical cues, they pay little or no attention to the accompanying figural cues. The most common solving strategy of this group is trial-and-error and the answer is usually provided by a recursive formula. As Rivera and Becker (2005, p. 199) stated, "a numerical mode of inductive reasoning uses algebraic concepts and operations (such as finite differences), whereas a figural mode relies on relationships that could be drawn visually from a given set of particular instances". The second group can be therefore characterized as predominantly figural generalizators. The solvers in this group are looking for figural clues and functional relationships among them. They usually use variables with meaning. In general, they can perceive relationships among the available cues, have a clear indication of how they interpret the figures drawn; they are better generalizators than numerical ones. Their answer is usually in an explicit formula.

As Lannin (2004, p. 217) stated, working with growing patterns could improve

> two important instructional goals for high school students: first, that they can reason flexibly, using recursive and explicit reasoning when faced with the need to create a mathematical

model for a situation; and second, that they recognize the advantages and limitations of these two ways of reasoning.

Lannin (2004) suggested a broad classification of the tasks into three categories: explicit-preferred tasks, recursive-preferred tasks and flexible tasks (both recursive and explicit forms are natural for students to provide). The all have their place in mathematics education, but we will use flexible tasks here.

2.2 Categories for analyzing the level of generalization

One of the possible ways to look at the level of generalization is through Piaget's Theory of cognitive development. Piaget (1980) identified four stages of cognitive development based on the age and skills of pupils. We have used his theory in our previous research (Slavíčková & Vargová, 2019) to identify changes in an abstraction of pupils who were on the edge of the concrete operational stage and formal operational stage. This analysis was not sensitive to small differences in pupils' cognitive development. Therefore, we have been looking for another way to analyze the pupils' and masters students' cognitive development. We need a tool that can be used objectively to explain pupils' and students' difficulties with a broad range of mathematical concepts and to suggest ways that they can use to learn these concepts. The system of structural development provided by Mulligan and Mitchelmore (2009) could be helpful in this. They define individual profiles of responses as one of four broad stages of structural development:

> Pre-structural stage: representations lack any evidence of mathematical or spatial structure; most examples show idiosyncratic features.
>
> Emergent (inventive-semiotic) stage: representations show some elements of structure such as the use of units; characters or configurations is first given meaning in relation to previously constructed representations.
>
> Partial structural stage: some aspects of mathematical notation or symbolism and/or spatial features such as grids or arrays is found.
>
> Stage of structural development: representations clearly integrate mathematical and spatial structural features. (Mulligan & Mitchelmore, 2009, p. 42)

These stages could be useful reference points to our analysis of pupils' and masters students' work. They are not dependent on the age of respondents, and only refer to their stage of the structural development within some mathematics concepts. Therefore, we will work with them in the *a priori analysis* of our test.

The *a priori analysis* is part of G. Brousseau's Theory of didactical situations (1997). This analysis is made before the teaching unit. It is a complex analysis of the teaching unit, and it should predict as accurately as possible its course. The *a priori analysis* can be characterized as an explanatory model of students' and teachers' behavior. Its goal is to identify potential obstacles, misconceptions, mistakes, corrections and further work with these mistakes. In this analysis, knowledge prerequisites necessary for the use of the different solving strategies are essential. After the teaching unit, an *a posteriori analysis* is made. In the a posteriori analysis, the *a priori analysis* is compared with experience from the realized teaching unit in the classroom, and recommendations for changes are formulated.

2.3 Research questions

Individuals tend to see the same pattern differently and produce different generalizations for it. There are several studies concerning pupils' strategies of generalization at primary level (Blaton & Kaput, 2005; Blanton, 2008; etc.) or secondary/middle school (Novotná et al., 2015; Eisenmann et al., 2017; Makowski, 2020; etc.) or looking for cognitive characteristics between middle school and primary school algebraic thinking (Rivera & Becker, 2005, 2007, 2008; etc.) Due to the age and skills of the target groups, linear patterns are usually posed to pupils. There have been several studies concerning the generalization of preservice mathematics teachers but most focus on elementary preservice mathematics teachers (Van Dooren et al., 2002, 2003; Hallagan et al., 2009; Strand & Mills, 2014; etc.).

In this article, we would like to continue that work and compare pupils' and masters students' methods of generalization. Therefore, we also add a non-linear pattern for finding the general formula. We stated three research questions:

> **RQ1**: Which strategies of generalization are used in solving flexible tasks by pupils aged 12–13?

RQ2: Which strategies of generalization are used in solving flexible tasks by prospective mathematics teachers?

RQ3: What are the main differences in preferred argumentation (or/and explanation) strategies depending on pupils' and students' knowledge and skills?

The third research question is closely related to the previous two. We posed this question to identify and summarize critical findings (or answers) for each group separately in the context of mathematical skills and knowledge. There are several reasons for looking at preferred argumentation (if any) in pupils' and masters students' work. Firstly, we use it as a support tool for analyzing the data to answer RQ1 and RQ2. Secondly, we are interested in getting a wider insight into pupils' ways of thinking. Thirdly, the data obtained could help us prepare an aspect of new research, including a teaching intervention.

3 Methodology

We worked with two groups of students. We posed them two flexible tasks for solving and arguing their findings/formulas. Both analysis *a priori* and analysis *a posteriori* were provided to find out answers to our research questions.

3.1 The sample

We worked with two different groups.

The first group comprised 32 pupils aged 12–13 in their first year at an 8-year grammar school in Bratislava. They encountered a variable as a representant of any number in a formula for finding the area of a square or a rectangle. In cooperation with their teacher, we prepared two flexible tasks to solve. Both tasks required pupils to use elements of mathematical structure such as numerical and geometrical patterns.

The second group comprised 19 prospective secondary mathematics teachers in the 1$^{\text{st}}$ year of their master's programme at the University.

Both groups were asked to explain their strategies of solving and generalizing. We were expecting a formal and age-appropriate way for this explanation and generalization. We were looking (separately in each group)

- whether they are predominantly numerical or figural generalizators,
- what stage of structural development according to Mulligan and Mitchelmore they were at.

3.2 The flexible tasks and additional questions

We posed two tasks, one on a linear pattern (Trapezoid table problem, inspired by Blanton, 2008) and one on an exponential one (a famous fractal – Sierpinski triangle). Both tasks are flexible; the recursive, and general formulas are easily formulated for them.

Task 1: A school has bought tables in the shape of trapezoids for the cafeteria. Two chairs are placed on the long side of the table and one chair is placed on each of the short sides. As shown below, the cafeteria staff put the tables end to end to save space. How many chairs can be placed around 3, 4, 10 and 50 tables?

Fig. 1: Trapezoid table problem illustration

Task 2: The Sierpinski triangle is constructed from an equilateral triangle by repeatedly removing smaller equilateral triangles. Start with an equilateral triangle (iter. 0), subdivide it into four smaller congruent equilateral triangles and remove the central triangle (iter. 1). Apply to each of the remaining smaller triangles (ad infinitum).

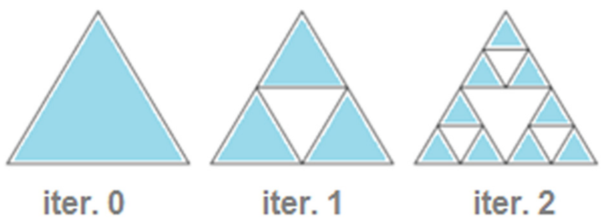

Fig. 2: Sierpinski triangle iterations

How many triangles will we have after repeating the process
a) 2 times (iter. 2), b) 3 times, c) 10 times, d) n-times?

Support data:
At the end, we asked pupils to answer additional questions concerning the tasks. We provided an on-line Google form with the following questions:
Question 1: Girl/Boy working on these tasks (mark one)
Question 2: How old are you?
Question 3a: Task one (Trapezoid table problem) was interesting/uninteresting
Question 3b: Task one (Trapezoid table problem) was for you: Easy to solve, Neither easy, nor difficult to solve, Difficult to solve
Question 3c: Justify your choice.
Question 4a: Task one (Sierpinski triangle) was interesting/uninteresting
Question 4b: Task one (Sierpinski triangle) was for you: Easy to solve, Neither easy, nor difficult to solve, Difficult to solve
Question 4c: Justify your choice.
Answers to these questions were used as additional data in the data analysis.

3.3 The data collection

The data collection was done in two lessons, one lesson per group. Both pupils and masters students worked on the aforementioned two tasks. The tasks were printed in grayscale; each pupil/student had a copy and was asked to write their workings/calculations on it. All were allowed to use calculators, rulers, pen/pencil, and as much paper as they needed (they could ask for more if needed, but no one used this option).

Due to the pupils' age, skills, and knowledge, we gave them 40 minutes for solving. We initially gave the masters students 20 minutes but several of them asked us for additional time. Finally, the students were given 30 minutes to solve the tasks.

After collecting the solutions, pupils and students were asked to answer the questions above. We assumed that 5 min would be sufficient, but in the end, we added extra minutes for pupils (due to their slow writing skills).

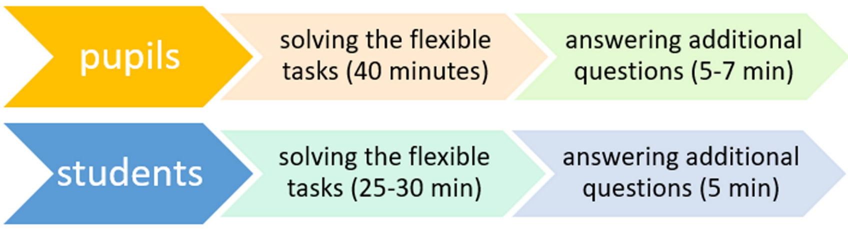

Fig. 3: Scheme of data collection in our groups

After collecting the data, students and pupils started to discuss the solutions. We monitored as observers. Unfortunately, there was not enough time at the lower secondary school for a more in-depth discussion (the data collection took longer than expected). Thus, the only supportive data we have are from the additional questions and observations.

3.4 The data analysis

Prior to assigning the tasks to pupils/students, we made part of the *a priori analysis* focus on possible strategies in the specific cognitive development of respondents first. The first task includes no variable and the far prediction is only supported by using a bigger number (50 tables in a row). The second task contains working with a variable on purpose – we wanted to know whether pupils would be able to use it in a more difficult context (they were not familiar with the powers of real numbers). Also, they could find a similarity between the tasks and therefore they could realize that the questions are practically the same.

Every provided solution was repeatedly read and analyzed in the context of the identified categories in Tab. 1. Comparison among both groups (pupils and students separately) and between them was made to find the answers to stated RQs. According to our theoretical review, we wanted to split the written solution into two main groups: figural and numerical solutions. We wanted to identify stages of structural development inside these groups according to Mulligan and Mitchelmore (2009) as described above. We stated the categories and assumed codes to be identified in the pupils' and students' solutions.

Tab. 1: Categories for data analysis

	Numerical	Figural
Pre-structural	Counting objects can be identified in the written solution	Drawing several pictures for smaller step numbers with a corresponding figural number
Emergent	Table of values (or similar representation of counting objects with a connection to step number) Some of the provided values do not correspond with the given figure (weak connection with the figure)	Drawings indicate essential figural elements: identifying elements the figure consist of (tables and chairs, triangles) The structure of finding the next figure is evident from their drawings
Partial structural	A written description of the recursive formula The solution does not indicate use of variable, but an understanding of numerical pattern is evident (for example, it is always bigger by 3 in the next step)	A clear idea of mechanism for higher step numbers, the suggestion of formula for bigger numbers (for example, it is multiple of step number)
Structural	Correct use of variable, formula is provided (mostly recursive)	Correct use of variable, formula is provided (mostly explicit)

Using these categories, we made the *a priori analysis* of expected strategies of generalization. To be clear about pupils' and masters students' ways of thinking and whether they were motivated to solve the tasks, we used supportive data from the questionnaire.

Question 3 (concerning the trapezoid table problem) and Question 4 (concerning the Sierpinski triangle problem) were basically the same. If respondents were interested in a specific problem, we assumed a higher effort to finish the solving. We also asked about the level of difficulty according to their opinion to find out how they perceived a specific problem. This additional data was used to decide why some (if any) tasks were not solved by pupils or students, how they perceived these kinds of tasks, and whether it was worth working with these kinds of tasks or whether another type of task would be more fruitful.

4 Results

First, the *a priori analysis* is presented. Next, the results concerning ways of generalizations are divided according to the research questions.

4.1 A priori analysis

According to our literature review, we expected two main strategies of generalization – numerical and figural. Both could be subdivided into several solving strategies as shown in Tab. 1.

Solving strategies which we expected from predominantly numerical generalizators are:

S1 (pre-structural): the pupil uses the operation and procedures which can "fit" this problem; the solution lacks any evidence of noticing a mathematical structure, mostly counting the objects in the figures is used, the general formula is missing or is invalid.

S2 (emergent): the solution shows some structure (a T-chart or some other way is used to show the correspondence between the step number and the value; no or incorrect symbolization is used.

S3 (partial structural): some aspects of mathematical notation or symbolism can be found; the general formula is given mostly in a recursive form with a clear understanding of the numerical pattern.

S4 (structural): algebraic representations integrate features of mathematical structural; the correct general formula is provided (correct in the sense of their thinking though they can make numerical errors).

The strategies for predominantly figural generalizators are similar:

S5 (pre-structural): the solution lacks any evidence of spatial structure, the figure is decomposed into the elements (or basic blocks), the correct solution for smaller values of step number.

S6 (emergent): the solution shows some elements of structure such as the identification of essential figural elements; the solution for specific numbers is provided (no symbolization is present)

S7 (partial structural): some aspects of spatial features such as grids or arrays can be found; the solution given in a recursive form with a clear understanding of the figural pattern is present.

S8 (structural): provided figures integrate mathematical and spatial structural features; the correct general formula with a clear reference to the figural cues is present.

4.2 Pupils' strategies of generalization and justification

To make our analysis clearer, we split it into two parts, one for a task.

4.2.1 Trapezoid table problem, pupils' strategies

We identified four groups of strategies of solving and generalization.

G1: taking one table with 4 chairs as a unit; thus, for n tables, there are $4n$ chairs

Pupils in this group did not take into account the overlap of the tables. They looked at the first table in a row and made their conclusion concerning the number of chairs.

G2: grouping tables by two and working with new units of two joined tables (see Fig. 4)

One of the solutions is: "The 'middle table' has 6 chairs. Then there are 2 tables at the ends, therefore plus two." (12-year-old boy) There are other 4 similar explanations.

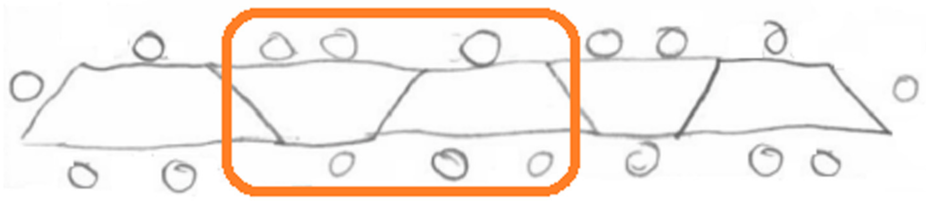

Fig. 4: Example of pupils' solution showing the "middle table"

In this group of solutions, pupils showed structural thinking – making groups of tables and constructing the solution for even numbers. It could be interesting to ask them for odd numbers to see whether they could fix the problem. Some pupils argued that if we had 50 tables, then we had to divide it by 2, then multiply by 3 and at the end add 2 (chairs). As in the G1 solution, we can identify a strong connection to the figure.

G3: decomposing and composing the structure: 5 chairs within one table, but after putting tables together, we have to remove 2 chairs; therefore, we are adding three chairs; only the recursive formula is provided

Pupils in this group showed structural thinking as they took a unit of one table and found out that we had to subtract overlapping chairs and work only with "the number of chairs at the bases of the trapezoids" and add two chairs at both ends of the table row. We can still identify the connection to the figure and the explanation within figural reasoning.

G4: counting "table by table": there are 4 chairs at the first table, 3 chairs at the 2^{nd}, 3^{rd}, 4^{th}, etc. table and 4 chairs at the last table

The decomposition strategy is used. The solution is clearly described, and the process of thinking can be easily identified. But the solution is on the low level of structural development, provided only for the specific numbers with an error for 50 tables.

Inductive reasoning and an indication of an infinite process can be identified here. Structural thinking is clearly present but most of the solutions had a numerical error. The question is whether the mistake was made due to carelessness or miscounting.

Our group of pupils at the secondary school rarely used a symbolic notation, they do not understand the principle of a variable yet. To explain their solving strategy, most of the pupils drew or described by words, how it works for a smaller number of tables and chairs. Some did not explain the solution (and provided the values for specific questions only).

4.2.2 Sierpinski triangle, pupils' strategies

We identified 3 groups differing in solving strategies and generalization for this task:

G5: Multiplying the previous result by 3, but the solution only works for small numbers (it does not work for the 10^{th} step, see Fig. 5)

The pattern was found, but as pupils were not familiar with the powers, it is not written with the notation of powers. The idea is clear, but the mathematical notation is not correct. The pupils who provided this type of solution and explanation (like in Fig. 5) are on a partial structural stage of their cognitive development.

G6: Misinterpretation of the task, counting all the triangles that could be found in the pictures with an increasing number of iterations

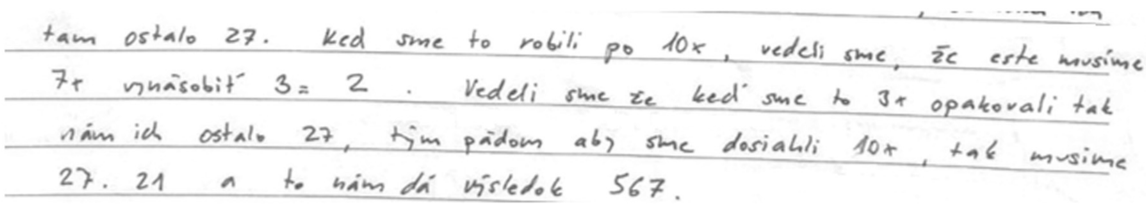

Fig. 5: Example of pupils' solution deriving the 10th power of 3[1]

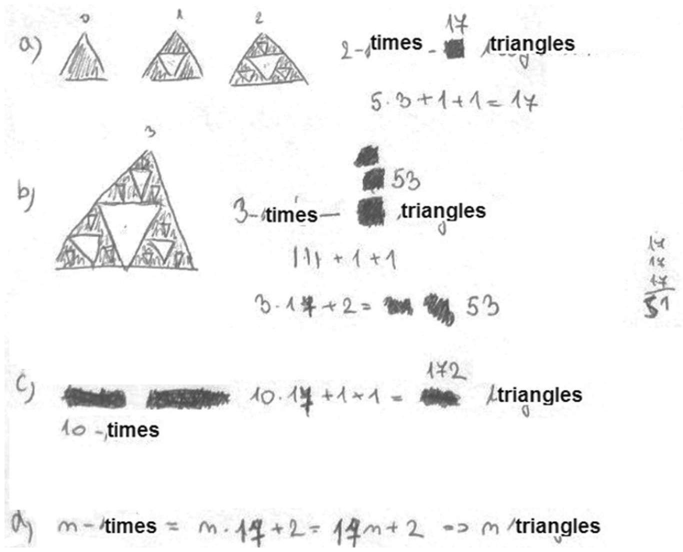

Fig. 6: Example of specific pupils' solution of task 2

Only one pupil provided this kind of solution (see Fig. 6). Some elements of structural thinking are present, therefore, the structural development is on the emergent stage.

G7: Correct solution with an understanding of repeated multiplication by 3

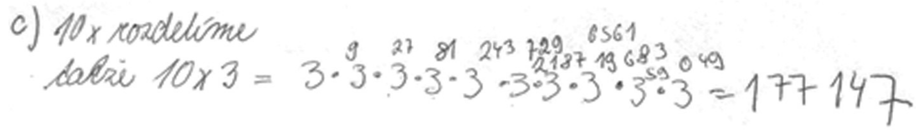

Fig. 7: Example of pupils' solution of Sierpinski triangle problem[2]

An explanation was mostly based on the figural interpretation. As we can see in Fig. 7, the notation $10 \times 3 = 3^n$ is not correct, but it is an adequate notation for the cognitive development and skills of the pupils in this age. As we can see later in the counting, the idea is correct, but the mathematical representation is missing. We can conclude that these pupils have good insight into the figure and its changing. According to the method of solving and the concepts used, they are at the partial structural stage.

4.3 Students' ways of generalization and justification

By analyzing the solutions of the pre-service mathematics teachers, we have found two main strategies of solving task 1 and five groups of strategies for solving task 2.

T1a: making a T-chart and/or corresponding $n \to a_n$, the result given in the general form

This solution indicates the numerical approach. The result is derived from the numerical cues and is mostly in general form for the n-th term.

T1b: only the solution without any explanation or showing the work

[1]Translation of solution: "When we did it 10-times, we knew that we had to 7-times multiply $3 = 2$ (erased). We knew that when we repeated it 3-times, it was 27. Therefore, to make it 10-times, we have to do multiplication 27 times 21 and it makes the result of 567."

[2]Translation: 10-times divided [the triangle], therefore $10 \times 3 = \ldots$

In this solution, we can only guess that students “looked and saw in the picture” the structure and how the pattern is changing. Therefore, we have put it in the graphical representation of students’ and pupils’ solutions with dash line (see Fig. 9).

T2a: general formula for the n-th term with figural reasoning

In this group, some students drew the 3rd iteration, but the majority put the answer from the first two iterations with the correct explanation by using figural reasoning. Clear structural development and the use of figural reasoning pointed to the structural development stage.

T2b: the correct solution raised from the numerical pattern (T-chart and/or corresponding $n \to a_n$) – numerical reasoning

In this group, the students made a numerical representation of the figure and derived the formula for the n-th term. Clear structural development and the use of numerical reasoning pointed to the structural development stage.

T2c: counting numbers of triangles inside the existing ones ($1 \to 1$, $2 \to 4$, $3 \to 9$, …, $n \to n^2$)

These students misinterpreted the task; they used the numerical approach and derived the general formula from numerical cues with no further reference to the figures.

T2d: counting all triangles possible found in a picture (similar to the pupils’ solution but without explanation or showing the calculations, a) 17, b) 53 and no solution for higher numbers or providing a general formula)

Another type of misinterpretation of the task can be identified in this group. The students have made the same calculations as the pupils did; however, they were “lazier”. They did it only for the first two sub-questions. Working exclusively with figures and counting the triangles pointed to the action level of structural development.

T2e: wrong interpretation of the task and a problem to write down the formula (recursive, or general one – Fig. 8)

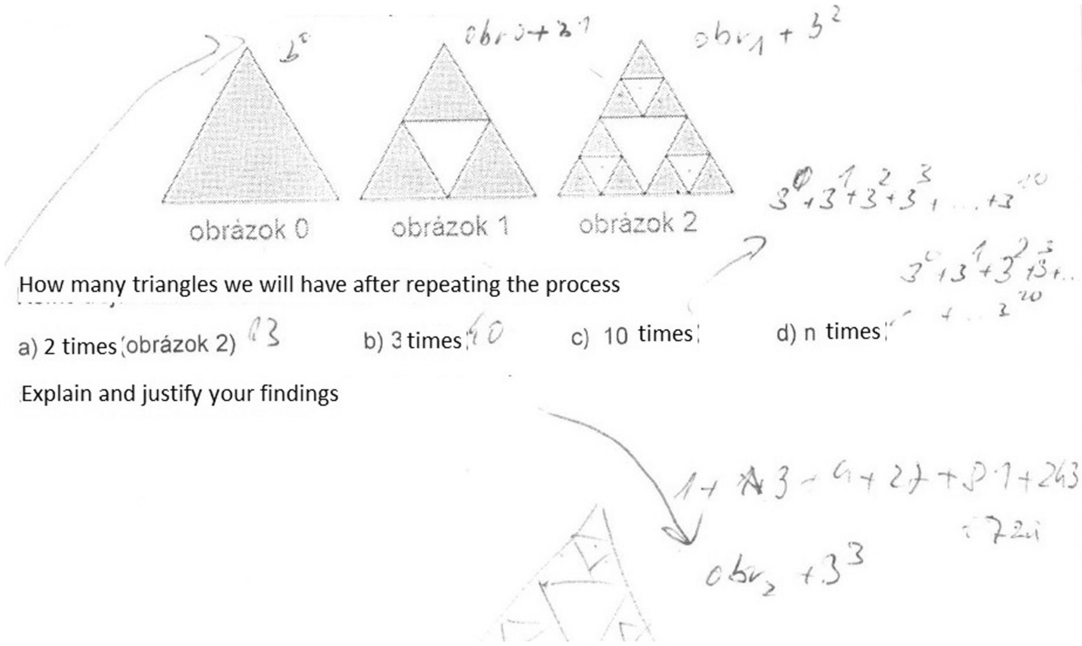

Fig. 8: Example of students’ solution

The cumulative sum of the results in the previous steps indicates two things. Firstly, the students wanted to use the mathematics they had just learned to do (sequences and sums). Secondly, they did not check the correctness of their numerical solution against the figure. They also counted with removed triangles. The answer showed students’ partial structured thinking; therefore, it corresponds with the emergent stage of structural development.

4.4 Differences within and between the groups

To answer the third research question, we have made a diagram of stated strategies in the analysis *a priori* (S1, …, S8) and strategies of generalization of the group of pupils (G1, …, G7) and prospective mathematics teachers (T’s).

Table 2 presents the comparison of results with our assumptions concerning strategies.

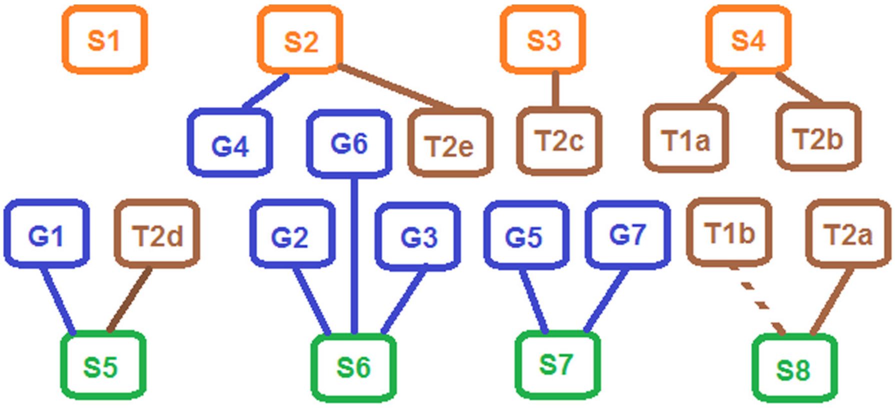

Fig. 9: Scheme of solving strategies and types of generalization (S1–S4 numerical, S5–S8 figural)

Tab. 2: Summary of findings

	Pupils of age 12–13	**Preservice mathematics teachers**
Strategy of generalization	Predominantly figural generalizators	Mostly numerical generalizators
Explanation	• explanation of the formula by showing on a figure • process of deriving the formula in a context of the given figure	• the given explanation is connected to numerical cues • figural cues are not crucial to the solution
Argumentation	rarely provided; showed on few examples that it works	not provided
Stage of structural development	mostly on an emergent inventive-semiotic or a partial-structural stage	some are on the basic level of generalization (pre-structural) or emergent inventive-semiotic level;
Solving strategies	several strategies of solving were identified, mostly trial-and-error, systematic experimentation and solution drawing	the trial-and-error strategy was present quite often
Effort	tried to do their best – solutions were mostly easy to read and comprehend	tried to use higher mathematics (like infinite sum)
Other	• reading literacy is a problem (several of them changed the meaning of the task and answered to a different problem/question) • low level of mathematical proficiency • several cases of low mathematical self-confidence (mostly in a group of pupils, we observed that they need their teacher's reassurance that their strategy is correct)	

Comparing the knowledge and skills of our two groups, we can conclude that the more experience they have, the more numerical strategies of generalization they use. Secondly, the university students tried to use scientific notation, and their way of expressing got shorter. Thirdly, none tried to justify his/her generalized findings. Therefore, for our groups of pupils and students, we can conclude, there are two main differences in the preferability of generalization strategy connected to their knowledge and skills. The first difference is in a generalization strategy (numerical or figural); the second one is operating with elements from the provided context (tables, triangles). A new question arises from this finding: Is it a natural change or our educational system has a strong influence on it?

5 Discussion

To answer RQ1, which strategies of generalization are used in solving flexible tasks by the pupils of age 12–13, we identified four approaches in solving Task 1 and three approaches in solving Task 2.

Answering RQ2, we observed that even some of the masters students have little knowledge of functions and writing recursive formula in the correct form. It is clear what they wanted to say by their formulas, but they failed in the formal notification. Similarly, Breiteig and Grevholm (2006) pointed out that "students prefer to explain in rhetoric rather than in symbolic algebra". Several studies (Rivera & Becker, 2007;

Richardson et al., 2009; Strand & Mills, 2014; etc.) indicate that pre-service mathematics teachers have strong procedural skills but they struggle to interpret and effectively use algebraic symbols, as well as to make connections between different representations.

Van Dooren et al. (2002, p. 322) found out that "nearly all students who wanted to become remedial teachers for primary and secondary education and about half of the future primary school teachers were unable to apply algebraic strategies properly or were reluctant to use them". In addition, a comparison of "secondary school preservice teachers at the beginning and the end of their teacher training showed that there were no differences in strategy use, mostly algebraic" (2002, p. 330). Master students in our study had similar problems to apply algebraic strategies properly. Therefore, changes in the preparation of future secondary mathematics teachers are necessary.

Rivera and Becker (2005) looked at how 42 middle school students performed inductive reasoning on two algebra tasks with growing patterns. Each task contained a sequence of figural and numerical cues. They found out that "middle school students have difficulty performing generalizations because many of those who teach them are predominantly more numerical than figural" (2005, p. 201). We did not provide numerical cues in our tasks. We did not want to affect pupils' and students' thinking and strategy of solving. Still, our findings are similar: our preservice students are taught mostly to do their conclusions using numerical cues (algebra, mathematical analysis, etc.).

We can conclude that both groups in our study are mostly predominantly figural generalizators; using the numerical strategy, they have a strong connection to the figure. Also, they are mainly on an emergent inventive-semiotic or a partial-structural stage. Similarly to Lee and Wheeler (1987), we found the dominance of manipulation over reasoning in the provided solutions; some respondents struggled to use algebraic symbolism in a meaningful way. Lee and Wheeler studied respondents' ability to recognize and express functional relationships (similar as we did by using dot patterns) and found out the same result as we did (see respondents' strategies G1, G5, G6, T2c, T2d, T2e). They conclude: "Seeing a pattern was not a problem, students lacked flexibility in generating sufficient possible patterns, in selecting useful ones, and in checking their validity." (Lee & Wheeler, 1987, p. 146)

Even though we asked the pupils and students to explain and justify the result, only a few of them did so. Pupils tried to describe their thinking and deriving formula. Some of them did not state a general claim, those who did so did not understand what it meant to justify that claim. It was not only the pupils who thought a few examples were sufficient to show that a general claim was valid. Similar results were observed in preservice mathematics teachers' solutions. Thus, we cannot really answer RQ3 about the main differences in preferred argumentation. Most pupils explained their findings but did not argue their correctness. Makowskis' (2020) results are close to our findings. She claims: "While the preservice mathematics teachers were willing to make a mathematical claim for patterns presented as numeric lists, they did not always have a justification." (p. 19)

The students in our sample are predominantly numerical generalizators, they pay a little attention to the figure, and the trial-and-error strategy is often used. However, as studies show, strategies of generalization can be learned. For example, Hallagan et al. (2009, p. 203) found out that prior to their intervention, students had difficulty expressing ideas in words, writing a generalization, recognizing patterns, and explaining a strategy, but they identified significant growth in their understandings of algebraic generalizations as a result of their intervention.

6 Conclusions, limitations and implications

As for the lower secondary pupils, we can sum up that their solutions showed a strong connection to the figure; they are mostly figural generalizators. Their attitude to solving the task was high. The context was appropriate to them, and we observed strong motivation during the whole time of solving, answering additional questions, and continuous discussion after the lesson ended. The most common question while solving the tasks was, "is it correct?". This indicates low confidence in the pupils' ability to solve a mathematical task. Therefore, we suggest that they should be asked to work on similar flexible tasks such as the ones we used in their mathematics lesson. At the same time, more attention to argumentation and generalization is needed.

On the other hand, the preservice mathematics teachers rarely checked the correctness of their numerical solution against the figure; they are primarily numerical generalizators. We know that pupils of age 13 do not have available tools for formal proof. We did not expect it; we hoped only for drawings, numerical expressions, and an oral explanation of their thinking. But preservice mathematics teachers do have tools for formal proof which they failed to use.

The study has several limitations. Firstly, there is the size of the groups that were given by the size of the classes. Secondly, there is a lack of time for interviewing respondents to obtain more profound feedback and more valuable data for our research. The reliance on only written solutions we consider as the biggest limitation of our study. Due to these limitations, we cannot claim general conclusions.

What do preservice teachers and others need to know to carry out instructions that support generalization and argumentation? To find an answer and build on it is the task for us as teacher educators in order to improve preservice mathematics teachers' education. Watson and Geest (2005, p. 228) suggest to provide tasks which establish working habits which may have been lost through disaffection and low expectation, develop routines of meaningful interaction, and be explicit about connections and differences in mathematics.

To sum up, more research is needed on the ability to generalize of pupils and future teachers at least in the countries whose curriculum does not include generalizing tasks. We suggest that such tasks are used in the preparation of future mathematics teachers so that their ability to generalize is enhanced and thus, they can develop this ability in their future pupils.

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